§1.3 - Classification of Differential Equations

**Ordinary**
\[
\frac{dy}{dx} = 3x + y \\
3 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0
\]

**Partial**
\[
\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 z}{\partial x^2} = 0 \\
\frac{\partial y}{\partial x} + \frac{\partial y}{\partial x_2} + \frac{\partial y}{\partial x_3} = 0
\]

**System**
\[
\begin{align*}
\frac{\partial z}{\partial x} &= 3x^2 - 3y \\
\frac{\partial z}{\partial y} &= -3x + 4y
\end{align*}
\]

**Definition**

Let \( f(x) \) be a function. An **ordinary differential equation** is an equation involving \( x, f(x), \) and some derivative(s) of \( f(x). \)

**Note**

- We typically replace \( f(x) \) by \( y, \) and often do not write \( y(x) \) explicitly, but \( y \) is a function of \( x. \)
- Nonetheless, the particular variable and function names \( (x, y, f) \) are unimportant, and you should understand their role from context. For example, while we won't usually do this, \( \frac{dx}{dy} = 5y \) is a perfectly valid ODE, as is \( x(f) + x'(f) = 3. \) (Though the latter is particularly obnoxious to read.)

**Definition**

The **order** of a differential equation is the order of the highest derivative

\[
\begin{align*}
\text{TE} & \rightarrow y' = 5y + x \quad \text{is order 1} \\
& \quad \downarrow \quad \downarrow \\
& \rightarrow y'' + y = 3 \quad \text{is order 2} \\
\text{TE} & \rightarrow y' y'' + y = x^2 \quad \text{is order 2} \\
& \quad \downarrow \\
& \rightarrow xy'' + 2y'' + (xy)^5 = x^3 \quad \text{is order 4}
\end{align*}
\]
A general $n^{th}$ order ODE has the form

$$G(x, y, y', ..., y^{(n)}) = 0$$

for some function $G : \mathbb{R}^{n+2} \to \mathbb{R}$. Necessary

1. $f(x)$ is an explicit solution for

$$G(x, y, y', ..., y^{(n)}) = 0$$
in an interval $I \subset \mathbb{R}$ if

$$G(x, f(x), f'(x), ..., f^{(n)}(x)) = 0 \quad \forall x \in I.$$

2. $y = x^2$ solves $(y'')^3 + (y')^2 - y - 3x^2 - 8 = 0$

(Why? Because if $f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2x^2 + (2x)^2 - x^2 - 3x^2 - 8 = 0$)

3. This is great news! It means that, although finding a solution may be challenging, checking if a particular function is a solution is easy. You can easily check your answer yourself for almost all the questions in this course!

### Linear vs. Nonlinear

1. An ODE

$$G(x, y, y', ..., y^{(n)}) = 0$$

is **linear** if \( \exists a_0(x), a_1(x), ..., a_n(x), b(x) \) such that

$$G(x, y, y', ..., y^{(n)}) = a_0(x)y + a_1(x)y' + ... + a_n(x)y^{(n)} + b(x).$$

It is **nonlinear** otherwise.

2. $3xy'' + x^2y = 5x$ is **linear**, $yy' = x$ is **nonlinear**.

We will see that solving linear ODEs is much easier.
Just as with equations, we may have systems of ODEs such as

\[
\begin{align*}
\frac{dx}{dt} &= ax + bxy \\
\frac{dy}{dt} &= cy + dxy
\end{align*}
\]

where the goal is to solve both ODEs simultaneously, that is we want functions \( f_1(t) \) and \( f_2(t) \) such that setting \( x = f_1(t) \) and \( y = f_2(t) \) makes \(^{(both)}\) the above equalities hold.